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SCATTERING FROM A PENETRABLE SPHERE AT HIGH FREQUENCIES

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Abstract

A scalar plane wave incident on a penetrable sphere is considered in the short wave length limit. A new representation of the scattering amplitude is introduced which is particularly appropriate in this limit, and which requires only the evaluation of certain integrals. Some of these may be evaluated asymptotically by the method of steepest descent and lead to the geometrical optics field contribution. Included in this is the bow field. The remainder of the integrals are evaluated by the method of residues and lead to the diffracted field contribution. This "diffracted ray" field is known from recent investigations in diffraction theory. An essential part of the analysis is the introduction of the parameter p, the number of internal refractions that a ray which hits the sphere undergoes. The results obtained are all in agreement with that which would be expected on the basis of geometrical diffraction theory.

I. Introduction

We consider a scalar plane wave incident on a penetrable sphere in the limit of high frequencies for which the wave length of the incident wave is small compared to the radius of the sphere. Quantum mechanically, the problem represents the scattering of a highly energetic particle by an attractive square well. It is also closely related to the corresponding electromagnetic problem which occurs in the scattering of light by water droplets. As such, it has engaged the attention of very many scientists whom we can not hope to enumerate here. A more or less complete historical record of the contributors to the raindrop problem may be found in van de Hulst's book [1]. The justification of the present work lies in the new representation of the scattering amplitude which is particularly appropriate to the study of it in the high frequency or short wave length limit. By means of it, it is possible to distinguish the contributions of the geometric optics field and of the diffracted field.

In a previous report [2] the forward scattering amplitude alone was considered. In the present study we consider the scattering amplitude at arbitrary angles. The basic idea utilized is the same as that of Rubinow and Wu [3] and Wu [4] in treating the forward scattering from impenetrable cylinders and spheres. It consists in converting the infinite series representation of the scattering amplitude into a sum of integrals. We then utilize an expansion first introduced by Debye in 1908 [5] in his work on the dielectric cylinder. In fact, it is worth remarking that the spirit of this work is no different from Debye's work. By means of Debye's expansion, it is possible to express the scattering amplitude $F(\theta)$ as a sum over p of partial amplitudes $F_p(\theta)$. The geometric meaning of p is that it represents the number of refractions that a ray which hits

the sphere undergoes. The partial amplitude $F_p(\theta)$ represents the contribution to the scattering at angle θ of the particular ray characterized by p refractions.

Each partial amplitude is represented by a sum of integrals. It is then possible to rearrange the integrals so that they split into two groups. In one group, the integrals have stationary points along their integration paths. In the second group, they do not. In making this split up, an idea introduced by Franz [6] is useful. The first group of integrals may be evaluated asymptotically by the stationary phase method or method of steepest descent. The result is called the geometric optics field because it is just what would be expected from purely geometric-optical considerations. The second group of integrals, called the diffracted field, is evaluated by the method of residues. They correspond physically to waves or rays which encircle the sphere and decay exponentially. Such phenomena are known from the work on impenetrable cylinders and spheres of Franz and Deppermann [7], where they are called "creeping waves". They are also known from the geometrical theory of diffraction of Keller [8], in which they are more appropriately named "diffracted rays". They occur in the scattering from smooth three dimensional objects [9] as well as from convex cylinders [10].

In this paper the language of geometrical diffraction theory is used throughout. Not only does this permit a ready physical identification of the various formulas, but likewise it provides a convenient physical check on the work. The rainbow phenomenon appears here as a result of a stationary point of the third order. For this reason, there is a greater intensity of the field at the bow angle. The bow angle is also a caustic of the p-family of rays, and is a local minimum or maximum of the scattering angle.

Yet to be examined is the field in the neighborhood of the axial caustics, i.e. in the forward and backward directions, as well as the field in the neighborhood of the boundary of the illuminated region associated with a given value of p.

II. Formulation

A scalar plane wave traveling in the direction $\theta = 0$ is incident on a penetrable sphere of radius a. The wave function ψ satisfies the scalar helmholtz equation

$$\left.\begin{aligned} \Delta\psi + N^2 k^2 \psi &= 0 \qquad r \leq a \\ \Delta\psi + k^2 \psi &= 0 \qquad r \geq a \end{aligned}\right\} . \tag{1}$$

The quantity N is called the refractive index of the sphere. The wave function and its normal derivative must be continuous at $r = a$. At large distances from the sphere, the wave function consists of the incident wave plus a scattered field:

$$\psi \sim e^{ikr \cos\theta} + f(\theta) \frac{e^{ikr}}{r} . \tag{2}$$

The function of interest $f(\theta)$ is called the scattering amplitude. As a convenience, we introduce the non-dimensional scattering amplitude

$$F(\theta) = f(\theta)/a \tag{3}$$

and non-dimensional parameter

$$x = ka . \tag{4}$$

The problem as defined represents the quantum-mechanical non-relativistic

scattering of a particle by a square-well potential. If the well is attractive, of depth V_o and range a, then the refractive index stands for

$$N = \sqrt{1 + 2mV_o a^2 / \hbar^2 x^2} \tag{5}$$

with m the mass of the particle and $\hbar$ equal to $(2\pi)^{-1}$ times Planck's constant. Nevertheless, we find it more descriptive to employ the language of geometric optics and diffraction theory. In other words, we think of this problem as representing the scattering of light from a dielectric sphere, idealized to the case in which the incident wave is a scalar. For simplicity, N will be assumed to be real (i.e., non-absorbing sphere) and greater than unity.

The scattering amplitude has the well known series representation [11]

$$F(\theta) = \frac{i}{x} \sum_{n=0}^{\infty} (n + 1/2) P_n(\cos\theta) \left[1 - e^{i2\delta(n)}\right] \tag{6}$$

where

$$e^{i2\delta(n)} = - \frac{Nh_n^{(2)}(x)j_n'(Nx) - h_n^{(2)'}(x)j_n(Nx)}{Nh_n^{(1)}(x)j_n'(Nx) - h_n^{(1)'}(x)j_n(Nx)}, \tag{7}$$

P_n is the Legendre polynomial of degree n, j_n, h_n are spherical Bessel, Hankel functions, respectively, and primes denote differentiation with respect to argument.

We seek the asymptotic form of $F(\theta)$ for large values of x. This is the neighborhood of the geometric optics limit for which the wave length $(2\pi/k)$ of the incident wave is small compared to the size of the scattering object. To examine this behavior, the representation (6) is not at all useful, because very many terms contribute to the sum, the number of such terms being of the order of x. We utilize instead a representation which essentially converts

the sum into an integral over the index. This conversion has been utilized successfully by Rubinow and Wu [3], and Wu [4], in finding the correction to the geometrical-optics scattering from impenetrable cylinders and spheres. The method is not unrelated to the method of the Watson transform[12], which has also been used extensively in recent years in the investigation of diffraction problems.

III The Scattering Amplitude Representation

We introduce the Fourier transform of the summand in equation (6). The summand is assumed to be defined for $n > -1/2$. By means of the Poisson summation formula [13], equation (6) becomes

$$F(\theta) = \frac{1}{x} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} d\nu \; \nu \; P_{\nu-1/2}(\cos\theta) \left[1 - e^{i2\delta(\nu-1/2)}\right] e^{i2\pi m(\nu-1/2)}. \qquad (8)$$

The function $P_{\nu-1/2}(\cos\theta)$ is to be understood as the hypergeometric function which is identical with the Legendre function only for integral values of ν. It is singular at $\theta = \pi$. Although the Poisson formula effects the conversion from summation over the discrete index n to integration over the continuous variable ν, an infinite sum remains, now taken over the new index m. However, it will be seen that the major contribution arises from the $m = 0$ term, just as for the conducting sphere problem. We desire to evaluate the integral for large x by the method of steepest descent. If the phase function $\delta(\nu-1/2)$ is examined as in reference [3], it will be found that the integrand in question has very many stationary points lying on the real axis. The integrand can be expanded in such a way that each term contains only one stationary point. In view of the relation

$$h_\nu(x) = \sqrt{\frac{\pi}{2x}}\ H_{\nu+1/2}(x) \tag{9}$$

for Hankel functions of either the first or second kind, we have

$$e^{i2\delta(\nu-1/2)} = \frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)} \left\{ R_{11} + T_{12}\, T_{21} \sum_{p=1}^{\infty} R_{22}^{p-1} \left[\frac{H_\nu^{(1)}(Nx)}{H_\nu^{(2)}(Nx)} \right]^p \right\} \tag{10}$$

where

$$\left.\begin{aligned}
R_{11} &= -\frac{U}{L} \\
R_{22} &= -\frac{V}{L} \\
T_{12} = 1 + R_{11} &= \frac{4i}{\pi x H_\nu^{(1)}(x) H_\nu^{(2)}(x) L} \\
T_{21} = 1 + R_{22} &= \frac{4i}{\pi x H_\nu^{(1)}(Nx) H_\nu^{(2)}(Nx) L} \\
U &= \frac{H_\nu^{(2)\prime}(x)}{H_\nu^{(2)}(x)} - N \frac{H_\nu^{(2)\prime}(Nx)}{H_\nu^{(2)}(Nx)} \\
V &= \frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)} - N \frac{H_\nu^{(1)\prime}(Nx)}{H_\nu^{(1)}(Nx)} \\
L &= \frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)} - N \frac{H_\nu^{(2)\prime}(Nx)}{H_\nu^{(2)}(Nx)}
\end{aligned}\right\} \tag{11}$$

This expansion was first introduced by Debye [5]. The quantities R_{11}, R_{22} are interpretable as Fresnel reflection coefficients for the interface (r = a) between the exterior (subscript 1) and the interior (subscript 2) of the sphere [14,15].

Thus, R_{11} represents the reflection coefficient of an inward spherical wave incident from the exterior and reflected back, while R_{22} represents the reflection coefficient for an outward spherical wave emanating from the center of the sphere and reflected back. The quantity T_{12} is the transmission coefficient for the wave moving from the exterior to the interior, and T_{21} is the transmission coefficient for the reverse process.

After inserting equation (10) into (8), we obtain

$$F(\theta) = \sum_{p=0}^{\infty} F_p(\theta) \ . \tag{12}$$

$$F_o(\theta) = \frac{i}{x} \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\nu \cdot \nu (-)^m \left[1 - \frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)} R_{11} \right] e^{i2\pi m\nu} P_{\nu-1/2}(\cos\theta). \tag{13}$$

$$F_p(\theta) = -\frac{i}{x} \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\nu \, f_{p,m} P_{\nu-1/2}(\cos\theta) \qquad p = 1,2,3,\ldots \ . \tag{14}$$

$$f_{p,m} = \nu (-)^m T_{12} R_{22}^{p-1} T_{21} \frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)} \left[\frac{H_\nu^{(1)}(Nx)}{H_\nu^{(2)}(Nx)} \right]^p e^{i2\pi m\nu}, \qquad p = 1,2,3,\ldots \ . \tag{15}$$

It is seen that $F(\theta)$ decomposes into parts labeled $F_o(\theta)$ and $F_p(\theta)$, representing, as we shall see, respectively, the contribution due to reflection from the surface of the sphere, and the contribution arising from (p-1) internal reflections, or p refractions in the sphere. From the present point of view, the expansion equation (10) has two advantages: it unfolds the stationary points of the integrand so that a given integral in the sum contains at most one station-

ary point, and it permits a ready identification of the multitude of terms in accordance with ray theory. However, the integrals are not yet in a proper form for evaluation because we must distinguish the contributions of the refracted rays and the diffracted rays. In order to make the separation, we first extend the integrals along the positive real ν-axis to the entire real ν-axis. To this end, we observe that

$$\left.\begin{aligned} H_{-\nu}^{(1)} &= e^{i\nu\pi} H_{\nu}^{(1)} & R_{11}(-\nu) &= R_{11}(\nu) \\ H_{-\nu}^{(2)} &= e^{-i\nu\pi} H_{\nu}^{(2)} & R_{22}(-\nu) &= R_{22}(\nu) \\ P_{-\nu-1/2} &= P_{\nu-1/2} & T_{12,21}(-\nu) &= T_{12,21}(\nu) \end{aligned}\right\} . \tag{16}$$

From these relations, it follows directly that

$$\int_0^{\infty} d\nu\, f_{p,m}\, P_{\nu-1/2}(\cos\theta) = \int_{-\infty}^{0} d\nu\, f_{p,p-m-1}\, P_{\nu-1/2}(\cos\theta) \quad \text{for p even.} \tag{17}$$

The terms with p odd have a slightly different reflection property about the origin of the ν-axis. To reveal it, substitute for $P_{\nu-1/2}(\cos\theta)$ which, from equation (84) of the appendix, is given by

$$P_{\nu-1/2}(\cos\theta) = -ie^{i\nu\pi} P_{\nu-1/2}(-\cos\theta) + (1 \pm e^{i2\pi\nu})\, Q_{\nu-1/2}^{(1)}(\theta) . \tag{18A}$$

The function $Q_{\nu}^{(1)}(\theta)$ is defined in equation (82). It is a linear combination of generalized Legendre functions of the first and second kind and is designed to have a purely exponential asymptotic form. In view of the sum over all values of m that occurs in $F_p(\theta)$, the second term on the right yields no contribution when the substitution for $P_{\nu-1/2}(\cos\theta)$ is made in $F_p(\theta)$. With the

remaining term in (18A) replacing $P_{\nu-1/2}(\cos\theta)$, it is clear that

$$\int_0^\infty d\nu f_{p,m} e^{i\pi\nu} P_{\nu-1/2}(-\cos\theta) = \int_{-\infty}^0 d\nu f_{p,p-m-2} e^{i\pi\nu} P_{\nu-1/2}(-\cos\theta) \text{ for p odd .} \quad (19)$$

Combining equations (17) and (19) with (14), and letting the summation over m commence with the value m = 0, we obtain

$$F_p(\theta) = -\frac{i}{x}(-)^{\frac{p+1}{2}} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\nu\, f_{p,m} e^{ip\pi\nu}\, iP_{\nu-1/2}(-\cos\theta), p = 1,3,5,\ldots , \quad (20)$$

$$F_p(\theta) = -\frac{i}{x}(-)^{\frac{p}{2}} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\nu\, f_{p,m} e^{ip\pi\nu} P_{\nu-1/2}(\cos\theta), \quad p = 2,4,6,\ldots . \quad (21)$$

In the same manner, we find for $F_o(\theta)$ that

$$F_o(\theta) = -\frac{i}{x}\left\{ i \sum_{m=0}^{\infty} (-)^m \int_{-\infty}^{\infty} d\nu\cdot\nu\, e^{i2\pi m\nu} e^{i\pi\nu} P_{\nu-1/2}(-\cos\theta) \right.$$

$$\left. + \sum_{m=0}^{\infty} (-)^m \int_{-\infty}^{\infty} d\nu\cdot\nu\, R_{11} \frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)} e^{i2\pi m\nu\cdot} P_{\nu-1/2}(\cos\theta) \right\} . \quad (22)$$

In order to separate off the contribution of the geometric optics field, we make use of an idea introduced by Franz [6]. Thus, we again utilize equation (18A) to substitute for $P_{\nu-1/2}(\cos\theta)$ or $P_{\nu-1/2}(-\cos\theta)$ into $F(\theta)$. In addition, the following alternative to (18A) will also be used which follows from (85) of the appendix.

$$P_{\nu-1/2}(\cos\theta) = ie^{-i\pi\nu} P_{\nu-1/2}(-\cos\theta) + (1 + e^{-i2\pi\nu}) Q^{(2)}_{\nu-1/2}(\theta) . \quad (18B)$$

When either of the substitutions given by equations (18A), (18B) is made in $F(\theta)$, it is found that only one term of the factor with $Q_{\nu-1/2}(\theta)$ survives the summation over m, contributed by the term m = 0:

$$F_p(\theta) = \frac{i}{x}(-)^{\frac{p-1}{2}} \left\{ \int_{-\infty}^{\infty} d\nu \, f_{p,o} \, e^{i(p-1)\pi\nu} \, (\pm) \, Q^{(1),(2)}_{\nu-1/2}(\theta) \right.$$

$$\left. \mp \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\nu \, f_{p,m} \, e^{ip\pi\nu} \, e^{\mp i\pi\nu} \, P_{\nu-1/2}(\cos\theta) \right\} ,$$

$$p = 1,3,5,\ldots , \qquad (23)$$

$$F_p(\theta) = \frac{i}{x}(-)^{\frac{p}{2}+1} \left\{ \int_{-\infty}^{\infty} d\nu \, f_{p,o} \, e^{i(p-1)\pi\nu} \, e^{\pm i\pi\nu} \, Q^{(1),(2)}_{\nu-1/2}(\theta) \right.$$

$$\left. \mp i \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\nu \, f_{p,m} \, e^{ip\pi\nu} \, e^{\pm i\pi\nu} \, P_{\nu-1/2}(-\cos\theta) \right\} ,$$

$$p = 2,4,6,\ldots , \qquad (24)$$

where the choices of sign are not independent, all upper signs belonging to the superscript (1) on $Q_{\nu-1/2}(\theta)$, and all lower signs belonging to the superscript (2). The upper sign corresponds to the use of equation (18A) and will be referred to as choice A. The lower sign corresponds to the use of equation (18B) and will be referred to as choice B.

In treating the reflected field in this manner, only choice A is utilized. There results

$$F_o(\theta) = F_o^G(\theta) + F_o^D(\theta) \ , \tag{25}$$

$$F_o^G(\theta) = -\frac{i}{x} \int_{-\infty}^{\infty} d\nu \cdot \nu \, R_{ll} \frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)} \, Q_{\nu-1/2}^{(1)}(\theta) \ , \tag{26}$$

$$F_o^D(\theta) = \frac{1}{x} \sum_{m=0}^{\infty} (-)^m \int_{-\infty}^{\infty} d\nu \cdot \nu \, e^{i2\pi m\nu} \, e^{i\pi\nu} \, P_{\nu-1/2}(-\cos\theta)$$

$$- \frac{1}{x} \sum_{m=0}^{\infty} (-)^m \int_{-\infty}^{\infty} d\nu \cdot \nu \, R_{ll} \frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)} \, e^{i2\pi m\nu} \, e^{i\pi\nu} \, P_{\nu-1/2}(-\cos\theta). \tag{27}$$

The first integral, involving $Q_{\nu-1/2}(\theta)$, is called the geometric optics field and is denoted by the superscript G. The remaining terms are called the diffracted field and are denoted by the superscript D. The justification for this terminology depends on the evaluation of the integrals and their geometric-optical identification. This will be accomplished in the next section. We do not yet label the geometric and diffracted field contributions for $F_p(\theta)$, because it turns out that firstly, both choices A and B must be made, depending on the numerical values of p and N. Secondly, it is necessary to make an s-fold substitution for $Q_{\nu-1/2}(\theta)$ from equation (18A) and (18B), again, depending on the values of p and N:

$$Q_{\nu-1/2}^{(1)}(\theta) = (-)^s e^{-i2\pi s\nu} Q_{\nu-1/2}^{(1)}(\theta) + \left[P_{\nu-1/2}(\cos\theta) e^{-i2\pi\nu} + i e^{-i\pi\nu} P_{\nu-1/2}(-\cos\theta) \right]$$

$$\times \sum_{\ell=0}^{s-1} (-)^\ell e^{-i2\pi\ell\nu} \ , \quad s = 1,2,3,\ldots \ , \tag{28A}$$

$$Q^{(2)}_{\nu-1/2}(\theta) = (-)^s e^{-12\pi s\nu} Q^{(2)}_{\nu-1/2}(\theta) + \left[P_{\nu-1/2}(\cos\theta) - ie^{-i\pi\nu} P_{\nu-1/2}(-\cos\theta) \right]$$

$$\times \sum_{\ell=0}^{s-1} (-)^{\ell} e^{-12\pi\ell\nu}, \quad s = 1,2,3,\ldots . \tag{28B}$$

Finally, then,

$$F_p(\theta) = F^G_p(\theta) + F^D_p(\theta) , \tag{29}$$

$$F^G_p(\theta) = \frac{i}{x}(-)^{\frac{p-1}{2}+s} \int_{-\infty}^{\infty} d\nu f_{p,o} e^{i(p-2s-1)\pi\nu} Q^{(1)}_{\nu-1/2}(\theta) \tag{30A}$$

$$F^D_p(\theta) = \frac{i}{x}(-)^{\frac{p+1}{2}+s} \left\{ i \sum_{\ell=0}^{s-1} \int_{-\infty}^{\infty} d\nu e^{i(p-2s)\pi\nu} f_{p,\ell} P_{\nu-1/2}(-\cos\theta) \right.$$

$$\left. + \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\nu e^{i(p-2s-1)\pi\nu} f_{p,m} P_{\nu-1/2}(\cos\theta) \right\} , \tag{31A}$$

choice A, $p = 1,3,5,\ldots$

or,

$$F^G_p(\theta) = \frac{i}{x}(-)^{\frac{p+1}{2}+s} \int_{-\infty}^{\infty} d\nu f_{p,o} e^{i(p-2s-1)\pi\nu} Q^{(2)}_{\nu-1/2}(\theta) \tag{30B}$$

$$F^D_p(\theta) = \frac{i}{x}(-)^{\frac{p+1}{2}+s} \left\{ i \sum_{\ell=0}^{s-1} \int_{-\infty}^{\infty} d\nu e^{i(p-2s)\pi\nu} f_{p,\ell} P_{\nu-1/2}(-\cos\theta) \right.$$

$$\left. - \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\nu e^{i(p-2s+1)\pi\nu} f_{p,m} P_{\nu-1/2}(\cos\theta) \right\} . \tag{31B}$$

choice B, $p = 1,3,5,\ldots$

$$F_p^G(\theta) = \frac{1}{x}(-)^{\frac{p}{2}+s+1} \int_{-\infty}^{\infty} d\nu e^{i(p-2s)\pi\nu} Q^{(1)}_{\nu-1/2}(\theta) f_{p,o} \qquad (32A)$$

$$F_p^D(\theta) = \frac{1}{x}(-)^{\frac{p}{2}+s} \left\{ \sum_{\ell=0}^{s-1} \int_{-\infty}^{\infty} d\nu e^{i(p-2s)\pi\nu} f_{p,\ell} P_{\nu-1/2}(\cos\theta) \right.$$

$$\left. + i \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\nu e^{i(p-2s+1)\pi\nu} f_{p,m} P_{\nu-1/2}(-\cos\theta) \right\} \qquad (33A)$$

choice A
p = 2,4,6,...

or,

$$F_p^G(\theta) = \frac{1}{x}(-)^{\frac{p}{2}+s+1} \int_{-\infty}^{\infty} d\nu e^{i(p-2s-2)\pi\nu} Q^{(2)}_{\nu-1/2}(\theta) f_{p,o} \qquad (32B)$$

$$F_p^D(\theta) = \frac{1}{x}(-)^{\frac{p}{2}+s} \left\{ \sum_{\ell=0}^{s-1} \int_{-\infty}^{\infty} d\nu e^{i(p-2s)\pi\nu} f_{p,\ell} P_{\nu-1/2}(\cos\theta) \right.$$

$$\left. - i \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\nu e^{i(p-2s-1)\pi\nu} f_{p,m} P_{\nu-1/2}(-\cos\theta) \right\} \qquad (33B)$$

choice B
p = 2,4,6,...

We see from equations (12), (13), (29), and (30) - (33), that the scattering amplitude is conveniently decomposed into a sum of fields $F_p(\theta)$. Each of these partial amplitudes is self-contained and is characterized by p, the number of refractions that a ray has undergone before contributing to the scattered field at the angle θ. It is also the number of straight line segments of a ray in the interior of the sphere. Each partial field consists of two

parts, a geometric optics contribution and a diffractive contribution. It suffices to know how to evaluate a typical partial field $F_p(\theta)$ to know how to evaluate all of them, although the reflected field $F_o(\theta)$ is a little different from the others. We are now in a position to evaluate the integrals appearing in equations (30) - (33).

IV. The Geometric Optics Field for the Reflected Ray

To evaluate a typical integral appearing in $F^G(\theta)$, we replace the Hankel functions appearing in the integrand by their Debye asymptotic form, and the $Q_{\nu-1/2}(\theta)$ functions by their Laplace asymptotic form, which are listed in the appendix, equations (88) - (91). After the above substitutions are made, the integral is evaluated asymptotically by the method of steepest descents. The criterion for the proper separation of the geometric optics field from $F(\theta)$ is just the existence of a saddle point. In this manner, the choice between A and B and the value of s can be determined. Only the major contribution of the saddle point evaluation will be retained.

Parenthetically, we remark that the use of the Debye asymptotic formulas always appears to be a good initial step in treating these types of integrals involving Hankel functions. Usually it is true that the major contribution of the integral does come from a region where the Debye approximation is valid.

Using equations (88) and (90) - (93) of the appendix, we obtain from equation (26)

$$F_o^G(\theta) \sim \frac{x^{\frac{1}{2}} e^{i\frac{\pi}{4}}}{\sqrt{2\pi \sin\theta}} \int_0^{\frac{\pi}{2}} \sin\tau \, (\cos\tau)^{\frac{1}{2}} \, \mathrm{Re}^{ix\phi_o} \, d\tau, \quad 0 < \theta < \pi , \qquad (34)$$

in which we have observed that

$$-R_{11} \sim R = \frac{N \sin \tau_1 - \sin \tau}{N \sin \tau_1 + \sin \tau} = \frac{\sin(\tau_1 - \tau)}{\sin(\tau_1 + \tau)} , \tag{35}$$

$$\left.\begin{aligned} \phi_o &= 2[(\tau - \theta/2) \cos \tau - \sin \tau] \\ \tau &= \cos^{-1}(\nu/x), \quad 0 \le \nu < x \\ \tau_1 &= \cos^{-1}(\nu/Nx) \end{aligned}\right\} . \tag{36}$$

The quantity R is the Fresnel reflection coefficient appropriate to the tangent plane at the point where the incident ray hits the sphere. See Figure 1, which also illustrates the geometrical interpretation of ν as an "impact parameter" measuring the distance of closest approach, in units of $1/k$, of the ray to the scattering center. The expression "impact parameter" is common to quantum mechanics, in which the incident plane wave is thought of classically as a stream of uniformly spaced particles moving orthogonally to the surfaces of constant phase of the plane wave. From equation (36), we see that the restrictions on ν imply that we consider here only those rays which impinge on the sphere (fig. 1).

The appearance of the large parameter x multiplying the phase function in the integrand in equation (34) is decisive in making the integrals a candidate for evaluation by the method of steepest descents (see the appendix). The condition for a saddle point is the vanishing of $d\phi_o/d\tau$, or

$$\frac{d\phi_o}{d\tau} = -2 \sin \tau(\tau - \frac{\theta}{2}) = 0 , \tag{37}$$

which yields the saddle point value (designated with subscript o)

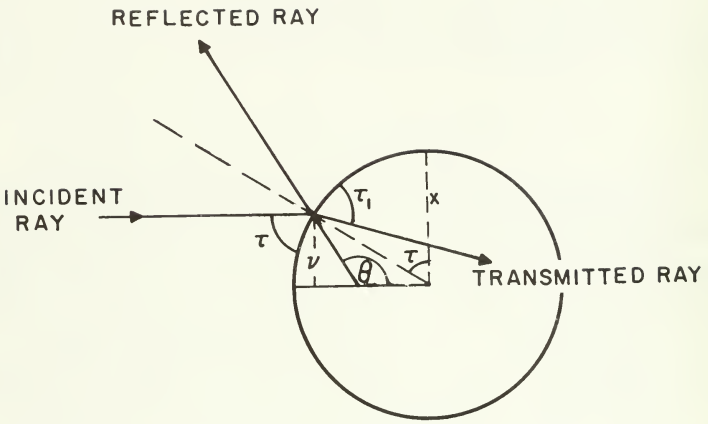

FIGURE 1

The incident ray hits the surface of the sphere and gives rise to a reflected ray and a transmitted ray. The radius of the sphere equals x, in units of the wave length = $2\pi/k$. The index ν measures the distance between the incident ray and the center of the sphere. At the impact point, the angles τ and τ_1 satisfy the Fresnel reflection formula.

$$\tau_o = \frac{\theta}{2}\ ,\quad \nu_o = x\cos\frac{\theta}{2}\ . \tag{38}$$

The solution $\tau = 0$ which arises from the vanishing of the factor $\sin\tau$ must be excluded because the Debye asymptotic forms lose their validity there. Furthermore, because of the restrictions of the Laplace asymptotic forms for $Q_\nu(\theta)$, the forward ($\theta = 0$) and backward ($\theta = \pi$) scattering directions must also be excluded. These directions are caustics of the family of reflected rays. The saddle point is of second order ($d^2\phi_o/d\tau \neq 0$), and according to equation (99), we obtain

$$F_o^G(\theta) \sim -\frac{1}{2}\,\frac{\sqrt{N^2 - \cos^2(\theta/2)} - \sin(\theta/2)}{\sqrt{N^2 - \cos^2(\theta/2)} + \sin(\theta/2)}\, e^{-2ix\sin(\theta/2)}, 0 < \theta < \pi\ . \tag{39}$$

This is the geometric optics externally reflected field. In the limit in which $N \to i\infty$, this reduces to the reflected field of an infinitely conducting sphere [7,16], as it should. In this case, the phase function of F_o^G is unaffected while the reflection coefficient R_{11} approaches -1, so that the amplitude of F_o^G is $(-1/2)$.

V. Geometric Optics Field for the Transmitted Rays

We proceed in the same manner for the p^{th} transmitted ray as given by (30A,B) and (32A,B). Then

$$F_p^G(\theta) \sim \frac{x^{\frac{1}{2}}(-)^{s+1}}{\sqrt{2\pi\sin\theta}} \int_0^{\frac{\pi}{2}} \sin\tau(\cos\tau)^{\frac{1}{2}}(1-R^2)R^{p-1}\exp\left(ix\phi_p(\pm\theta)\pm i\,\frac{\pi}{4}\right)d\tau\ ,\ p=1,3,5,\ldots\ . \tag{40}$$

$$F_p^G(\theta) \sim \frac{x^{\frac{1}{2}}(-)^{s}}{\sqrt{2\pi\sin\theta}} \int_0^{\frac{\pi}{2}} \sin\tau(\cos\tau)^{\frac{1}{2}}(1-R^2)R^{p-1}\exp\left(ix\phi_p\left[\pm(\pi-\theta)\right]\pm i\,\frac{\pi}{4}\right)d\tau, \tag{41}$$

$$p=2,4,6,\ldots\ ,$$

since $R_{22} \sim R$. The upper and lower signs in the exponent go together. The upper signs correspond to choice A, while the lower ones correspond to choice B. Although the limits of integration in equations (40) and (41), as well as in (34), extend over a finite interval, there are no end-point corrections to the saddle point evaluation we are making, in spite of the restrictions of equations (36). This is because the original integrals extended over the entire ν-axis. The phase function ϕ_p is defined by

$$\phi_p(\pm\theta) = 2\left\{pN\sin\tau_1 - \sin\tau - \left[p\tau_1 - \tau - (p-2s-1)\frac{\pi}{2} \mp \frac{\theta}{2}\right]\cos\tau\right\}, \quad p = 1,3,5,\ldots, \tag{42}$$

$$\phi_p\left[\pm(\pi-\theta)\right] = 2\left\{pN\sin\tau_1 - \sin\tau - \left[p\tau_1 - \tau - (p-2s-1)\frac{\pi}{2} \mp (\pi-\theta)/2\right]\cos\tau\right\}, \quad p = 2,4,6,\ldots. \tag{43}$$

The condition for the existence of a saddle point for these phase functions, $d\phi_p/d\tau = 0$, is then

$$\theta = \mp\left[-\pi + 2p\left(\frac{\pi}{2} - \tau_{10}\right) + 2\tau_o - 2s\pi\right], \quad p = 1,3,5,\ldots, \tag{44}$$

$$\theta = \pi \pm \left[-\pi + 2p\left(\frac{\pi}{2} - \tau_{10}\right) + 2\tau_o - 2s\pi\right], \quad p = 2,4,6,\ldots, \tag{45}$$

inasmuch as

$$\frac{d\phi_p(\theta)}{d\tau} = 2\sin\tau\left[p\tau_1 - \tau - (p-2s-1)\frac{\pi}{2} - \frac{\theta}{2}\right]. \tag{46}$$

Assuming the saddle point is of second order, the dominant contribution to $F_p^G(\theta)$ is

$$F_p^G(\theta) \sim (-)^{p+s}\left[\sin\theta\phi_p''(\tau_o)\right]^{-\frac{1}{2}}\sin\tau_o(\cos\tau_o)^{\frac{1}{2}}R^{p-1}(1-R^2)\exp\left(i\left[x\phi_p(\tau_o) \pm \frac{\pi}{4}\right] + i\frac{\pi}{4}\right),$$

$$p = 1,2,3,\ldots, \tag{47}$$

where, from (46),

$$\phi''_p(\tau_o) = \frac{d^2\phi_p(\tau_o)}{d\tau_o^2} = 2 \sin \tau_o \left(\frac{p \sin \tau_o}{N \sin \tau_{10}} - 1 \right) , \tag{48}$$

$$\left.\begin{aligned} \phi'_p &= \phi_p(\pm\theta) && \text{if p is odd} \\ \phi'_p &= \phi_p[\pm(\pi-\theta)] && \text{if p is even} \end{aligned}\right\} , \tag{49}$$

and τ_o is the solution of the transcendental equation (44) or (45). The sign in front of the factor $\frac{\pi}{4}$ in the exponent in (47) agrees with the sign in $\phi'_p(\tau_o)$. It has been assumed above that the amplitude part of the integrand behaves like a constant in the neighborhood of the saddle point. Because R is a number which is always less than unity, the larger values of p are of decreasing importance because of the quantity R^{p-1} which appears. The latter is just the attentuation factor due to the repeated internal reflections.

It remains to specify the choice of sign in the argument of ϕ'_p as well as the value of s. The criterion for this is just that the choice should yield a saddle point, i.e., equations (44) or (45) should yield a solution $\theta(\tau_o)$ such that $0 < \theta < \pi$ and $0 < \tau_o \leq \frac{\pi}{2}$. It is easy to solve the saddle point equation explicitly only for the case p = 1. Then we must choose s = 0 and selection A, when the value of the saddle point is

$$\tau_o = \cos^{-1} \frac{N \sin(\theta/2)}{\sqrt{N^2 - \frac{2}{N} \cos(\theta/2) + 1}} , \tag{50}$$

or

$$\nu_o = x \frac{N \sin(\theta/2)}{\sqrt{N^2 - \frac{2}{N} \cos(\theta/2) + 1}} . \tag{51}$$

From (50), we see that the greatest angle of refraction occurs for $\tau_o = 0$, or

$$\theta = 2\cos^{-1}(1/N) \ . \tag{52}$$

This angle is therefore the boundary of the family of rays associated with $p = 1$ (see Figure 2). Because of its association with $\tau = 0$, the angle at which the Debye formulas become invalid, it must be excluded from the range of applicability of the partial amplitude $F_1^G(\theta)$. From equation (47), the contribution of the directly transmitted ray, or $F_1^G(\theta)$, is

$$F_1^G(\theta) \sim -\left[1 - \left(\frac{N^2-2N\cos(\theta/2)+1}{N^2-1}\right)^2\right] \frac{N\sqrt{\sec(\theta/2)\left[N\cos(\theta/2)-1\right]\left[N-\cos(\theta/2)\right]}}{2\left[N^2-2N\cos(\theta/2)+1\right]}$$

$$\times \exp\left(12x\sqrt{N^2-2N\cos(\theta/2)+1}\right) \ , \ 0 < \theta < 2\cos^{-1}\frac{1}{N} \ . \tag{53}$$

Except for the factor in square brackets which has its origin in the transmission loss that occurs when the ray enters and leaves the sphere, this result agrees with the differential cross section for the scattering by an attractive square well as calculated according to the trajectories of classical mechanics [17]. Naturally, the phenomenon of partial reflection is alien to classical mechanics so that the difference of the factor $(1-R^2)$ must occur.

The saddle point evaluations for F_R^G and F_1^G are essentially the same as those made by Bucerius [18] in his approximate treatment of the scattering of light by a water droplet. For arbitrary p, it is not possible to exhibit τ_o as an explicit function of θ. Nevertheless, an important property of equations (44) and (45) can be stated: there is always a saddle point at $\tau_o = \pi/2$ when $s = 0$. This value for the saddle point ($\nu_o = 0$) corresponds to the ray

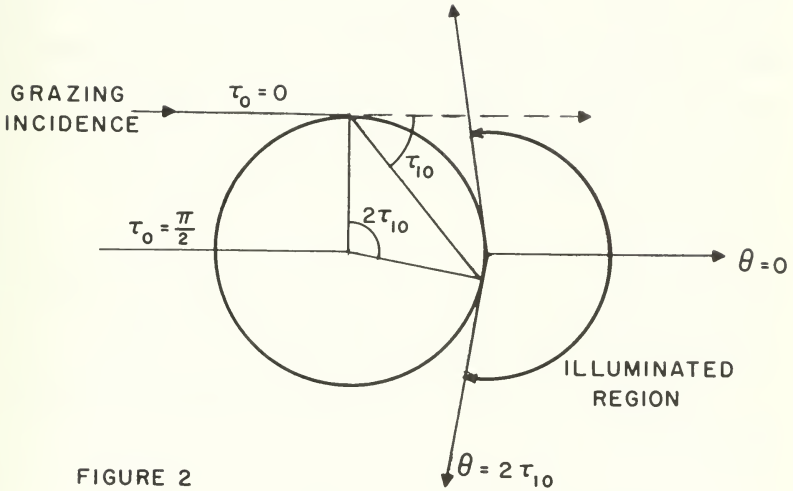

FIGURE 2

The directly transmitted rays, associated with the value $p = 1$, are confined to the illuminated region $0 < \theta < 2\tau_{10}$ where $\tau_{10} = \cos^{-1}\frac{1}{N}$. The extremum values occur, from equation (44), for $\tau_o = 0$ and $\tau_o = \pi/2$. The parameter s must be chosen equal to zero.

which hits the center of the sphere. For this value, $\tau_{10} = \pi/2$, and p odd requires $\theta = 0$, while p even requires $\theta = \pi$. This means that, for odd p, there is always some neighborhood of the forward direction which is illuminated. In geometric optics, it is usual to call this the lit region. In the present context, lit region means the region of accessibility of the family of rays associated with a given value of p. This region is different for different values of p. Similarly, for p even, there is always some neighborhood of the backward direction which is a lit region. In short, the lit region for odd-p rays is essentially forward, and for even-p is essentially backwards.

The saddle point equations, (44) and (45), for $p \geq 2$, can be studied in a qualitative manner in order to decide the choice of equations A or B and the value of s for F_p^G. We shall use the notation (s,A) or (s,B) to denote a particular selection. Let us examine first the selection (s,A) for even p. In the neighborhood of τ_o equal to zero or $\pi/2$, we find from equation (45) with the upper sign that

$$\left.\begin{aligned} \theta &= 2p \sin^{-1}\frac{1}{N} - 2\pi s + \epsilon + \ldots \qquad \text{when } \tau_o = \epsilon \\ \theta &= \pi - 2\pi s + 2\left(1 - \frac{p}{N}\right)\epsilon + \ldots \qquad \text{when } \tau_o = \frac{\pi}{2} - \epsilon \end{aligned}\right\} . \tag{54}$$

Furthermore,

$$\frac{d\theta}{d\tau_o} = 0 \qquad \text{for } \tau_o = \tau_B \equiv \sin^{-1}\sqrt{\frac{N^2-1}{p^2-1}} \quad . \tag{55}$$

These relations are sufficient to describe the qualitative behavior of θ as a function of τ_o for various values of N and p. Thus, for $N > p$, θ is a monotonic

increasing function of τ_o. For $N < p$, θ increases from its value $(2p \sin^{-1}\frac{1}{N} - 2\pi s)$ to a maximum at τ_B, and then decreases to the value $(\pi - 2\pi s)$ at $\tau_o = \pi/2$. This behavior is illustrated schematically for (0,A) and $p = 4$ in Figure 3. The condition that θ be restricted to $0 < \theta < \pi$ now produces the following selections. For $N > p$, clearly the choice (0,A) is to be made. As N decreases below p, values of $\theta > \pi$ would appear for (0,A). Consequently we must choose instead (0,B). This has the effect of reflecting the $\theta(\tau_o)$ curve for (0,A) in the line $\theta = \pi$, and is suitable for $\csc\frac{\pi}{2p} \leq N \leq p$. For $N < \csc\frac{\pi}{2p}$, $\theta(\tau_o)$ for (0,A) becomes larger than 2π. This requires the selection of (1,A). As can be seen from equations (54), this has the effect of reducing θ by 2π. This choice applies to the neighborhood of $\tau_o = 0$. In the neighborhood of $\tau_o = \pi/2$, we must always end up with $s = 0$. In general, for p large and N close to unity, a succession of selections occur as τ_o varies from zero to $\pi/2$. The sequence commences with (s,A) or (s,B), perhaps increases in s-value, but ultimately decreases and concludes with (0,B). A typical sequence might be $\Big\{(s,A), (s,B), (s,A), (s\text{-}1,B), (s\text{-}1,A)\ldots, (1,A), (0,B)\Big\}$. In Table I, we summarize the selections as a function of N for $p = 2$. In this case, we must take $s = 0$ for all values of N. The lit regions are also indicated in the table. For $p = 4$, the selections for various ranges in value of N are indicated schematically in Figures 3I - 3V. These figures correspond to the curves labeled I - V in Figure 3.

The consideration of equation (44) for odd p is quite similar. We have seen that for $p = 1$, too, the selection (0,A) is required for all values of N. To illustrate the situation for larger values of p, we have plotted the saddle point equation (44) in Figure 4 in a schematic manner, for $p = 3$, $s = 0$, and selection A, for a few values of N. The reduction to the interval $0 < \theta < \pi$

TABLE I

N	τ_o	Illuminated or lit region	Selection
$2 \leq N \leq \infty$	$0 < \tau_o \leq \frac{\pi}{2}$	$4 \sin^{-1} \frac{1}{N} < \theta < \pi$	(0,A)
$\csc \frac{\pi}{4} \leq N \leq 2$	$0 < \tau_o \leq \tau_o'$	$4 \sin^{-1} \frac{1}{N} < \theta < \pi$	(0,A)
	$\tau_o' \leq \tau_o \leq \frac{\pi}{2}$	$\theta(\tau_B) < \theta < \pi$	(0,B)
$1 \leq N \leq \csc \frac{\pi}{4}$	$0 < \tau_o \leq \frac{\pi}{2}$	$\theta(\tau_B) < \theta < \pi$	(0,B)

Table I. The selections of (0,A) or (0,B) for varying values of N for the geometric optics field for p = 2. The value τ_o' is that value of $\tau_o < \tau_B$ for which $\theta(\tau_o)$ equals π, as given by equation (45) for p = 2 and s = 0. The angle $\tau_B = \sin^{-1}[(N^2-1)/3)]^{1/2}$ is defined in equation (55).

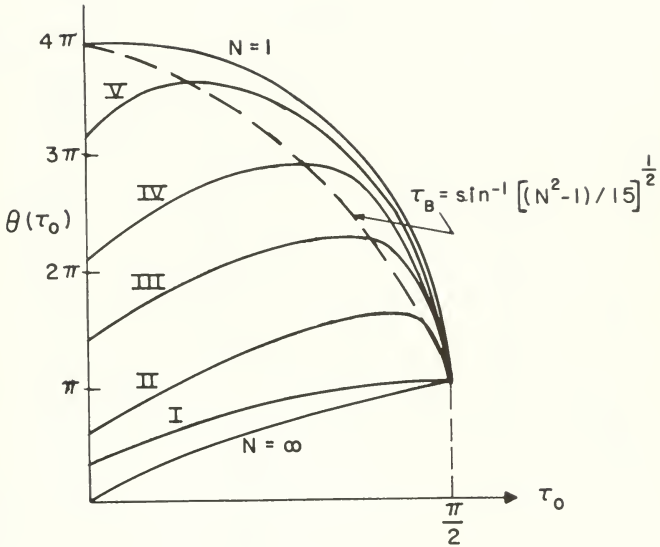

FIGURE 3

The scattering angle θ as a function of τ_o as determined by equation (45). The curves are schematic and not literal, for $p = 4$, $s = 0$, and selection A (upper choice of sign). A succession of values of N are shown, varying from $N = \infty$ to $N = 1$. The dotted curve labeled τ_B is the locus of points for which $d\theta/d\tau_o = 0$. The $\tau_o = 0$ intercept is $\theta = 8\sin^{-1}\frac{1}{N}$.

The $\theta(\tau_o)$ curves for $p = 4$ labeled I through V in Figure 3 are reduced to the physical interval $0 < \theta < \pi$ by means of the proper choice of the value of s (reduction by 2π) and of the choice A or B (reflection in the line $\theta = \pi$). The range of values of N appropriate to each selection is indicated below each figure.

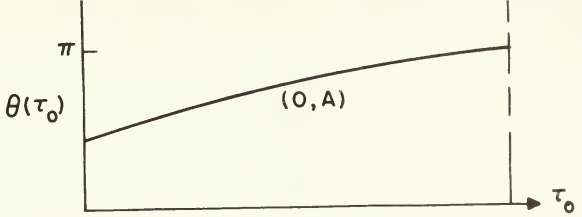

FIGURE 3I: $4 \leq N < \infty$

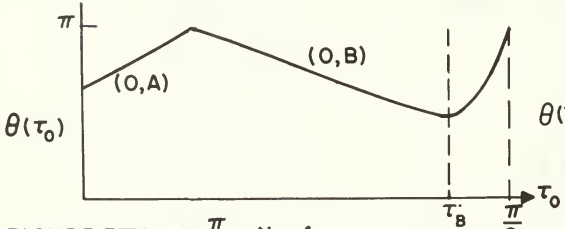

FIGURE 3II: $\csc \frac{\pi}{8} \leq N \leq 4$

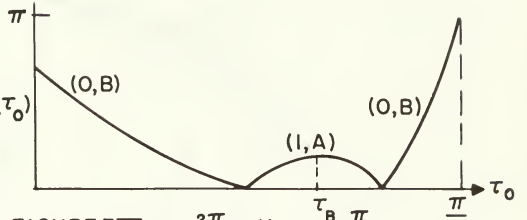

FIGURE 3III: $\csc \frac{2\pi}{8} \leq N \leq \csc \frac{\pi}{8}$

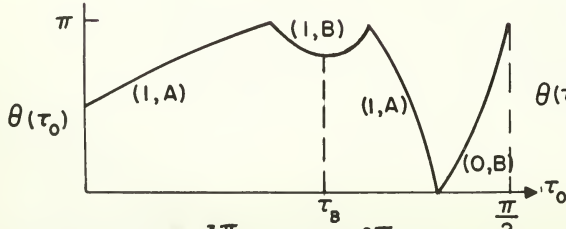

FIGURE 3IV: $\csc \frac{3\pi}{8} \leq N \leq \csc \frac{2\pi}{8}$

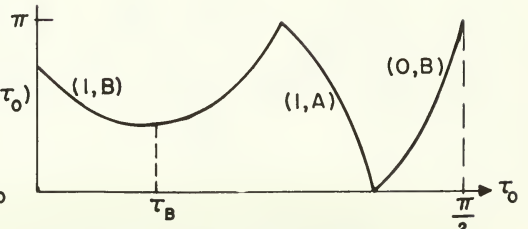

FIGURE 3V: $1 \leq N \leq \csc \frac{3\pi}{8}$

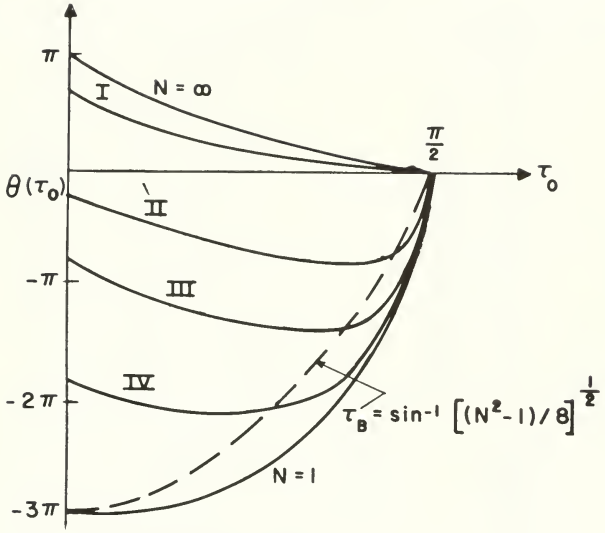

FIGURE 4

Schematic illustration of equation (44) for p = 3, s = 0, selection A (upper choide of sign), and a succession of values of N from N = ∞ to N = 1. The dotted curve τ_B is the locus of points for which $d\theta/d\tau_o = 0$. The $\tau_o = 0$ intercept is $\theta = \pi - 6 \sin^{-1}(1/N)$.

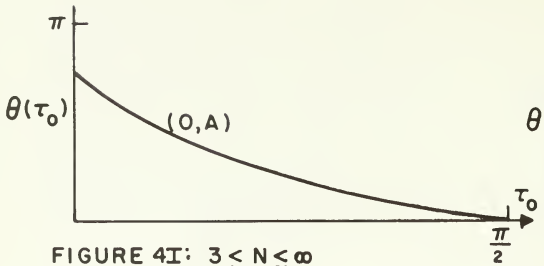

FIGURE 4I: $3 \leq N \leq \infty$

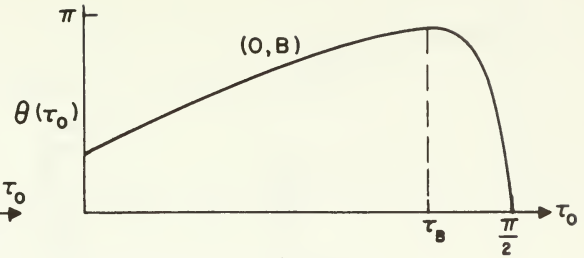

FIGURE 4II: $\csc \frac{\pi}{6} \leq N \leq 3$

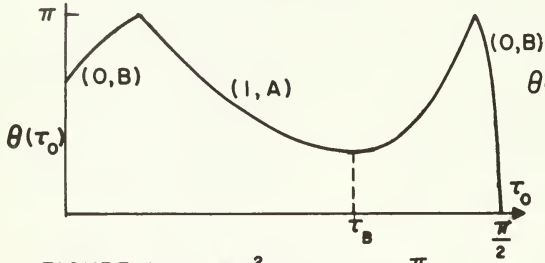

FIGURE 4III: $\csc \frac{2}{6}\pi \leq N \leq \csc \frac{\pi}{6}$

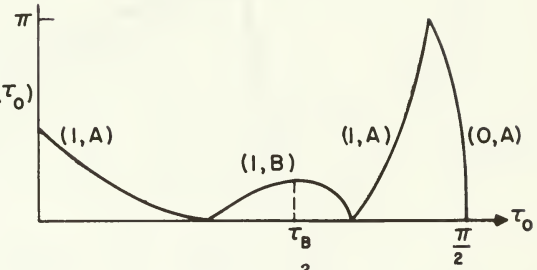

FIGURE 4IV: $1 \leq N \leq \csc \frac{2}{6}\pi$

The $\theta(\tau_0)$ curves for p = 3 labeled I through IV in Figure 4 are reduced to the physical interval $0 < \theta < \pi$ by means of the proper choice of the value of s and of the choice A or B. The values of N appropriate to each selection are indicated below each figure.

via the correct sequence of selections is then indicated in Figures 4I - 4IV. The values of N for which each selection is appropriate are also given. As soon as a value of s is utilized which is ≥ 1, the lit region extends over all possible scattering angles. The geometric significance of the parameter s is just this: by means of p refractions in the interior of the sphere, the p-ray is able to encircle the origin s times. The total angle of refraction is then, so to speak, $\Theta + 2\pi s$. This is the reason for the utilization of equations (28A) and (28B). For $p = 3$ and $p = 4$, the largest value of s that is needed is $s = 1$. It is almost superfluous to add that the appropriate form of equation (47) must always be utilized to obtain $F_p^G(\Theta)$.

VI. Bows

It sometimes may happen that the saddle point is of third order, i.e., both ϕ_p' and ϕ_p'' vanish at some point. This causes a greater intensity to be produced at certain special angles, and is the explanation of the familiar rainbow that occurs in the scattering of light by water droplets. In analogy with this phenomenon, we will call the geometric fields associated with third order saddle points bows. If we impose the condition that ϕ_p'' given by equation (48) vanish, we see that the saddle point does not now occur for every value of τ_o in the interval $(0,\pi/2)$, but rather only at the special value τ_B, where

$$\left.\begin{aligned} \tau_B &= \sin^{-1}\sqrt{\frac{N^2-1}{p^2-1}} \\ \text{or} \qquad \nu_B &= x\sqrt{\frac{(p^2-1)^2-(N^2-1)^2}{p^2-1}} \end{aligned}\right\} \quad p > 1 \tag{56}$$

The bow does not occur for p = 1. By comparison with equation (55), we see that the bow angle θ_B at which the bow occurs is always a scattering angle of local minimum or local maximum deviation. Therefore, it is also a caustic for the set of p-rays. Furthermore, for a given value of N, the bow occurs only for those values of p for which

$$p \geq N \ . \tag{57}$$

The most intense bow is naturally the one with the smallest value of p which satisfies the above relation, because of the attenuation factor R^{p-1}. The bow angle is obtained from (44) or (45) with $\tau_o = \tau_B$. At the bow angle, it follows from equation (100) of the appendix that the following expression appropriate to third order saddle points replaces (47).

$$F_p^G(\theta_B) \sim (-)^{p+s} x^{\frac{1}{6}} (6^{\frac{1}{3}}\pi \sin\theta_B)^{-\frac{1}{2}} \Gamma(\tfrac{1}{3}) \left\{ \sin\tau (\cos\tau)^{\frac{1}{2}} R^{p-1} (1-R^2) \left[\phi_p'''(\tau)\right]^{-\frac{1}{3}} \right.$$

$$\left. \exp\left(i\left[x\phi_p(\theta_B) \pm \tfrac{\pi}{4} \right] \right) \right\}_{\tau=\tau_B} \quad , \quad p = 2,3,4,\ldots \ , \tag{58}$$

where

$$\theta_B = \begin{cases} \mp\left[-\pi + 2p\sin^{-1}\left(\dfrac{(p^2-N^2)^{1/2}}{N(p^2-1)^{1/2}} \right) + 2\sin^{-1}\left(\dfrac{N^2-1}{p^2-1} \right)^{\frac{1}{2}} - 2s\pi \right], & p = 3,5,7,\ldots \\ \pi \pm \left[-\pi + 2p\sin^{-1}\left(\dfrac{(p^2-N^2)^{1/2}}{N(p^2-1)^{1/2}} \right) + 2\sin^{-1}\left(\dfrac{N^2-1}{p^2-1} \right)^{\frac{1}{2}} - 2s\pi \right], & p = 2,4,6,\ldots \ , \end{cases} \tag{59}$$

$$\phi_p(\tau_B,\theta_B) = 2(p^2-1)^{\frac{1}{2}}(N^2-1)^{\frac{1}{2}}, \qquad p = 2,3,4,\ldots, \quad (60)$$

$$\phi_p'''(\tau_B,\theta_B) = \frac{2p(N^2-1)\sin\tau_B\cos\tau_B}{N^3\sin^3\tau_{1B}} = \frac{2}{p^2}(p^2-1)^{\frac{1}{2}}(p^2-N^2)^{\frac{1}{2}}, \quad p = 2,3,4,\ldots, \quad (61)$$

$$R(\tau_B) = \frac{p-1}{p+1}, \qquad p = 2,3,4,\ldots. \quad (62)$$

The signs in front of $\frac{\pi}{4}$ in (58) agree with the signs in (59). In contrast with $F_p^G(\theta)$ as given by (47), which is of order unity, the bow term as given by (58) is of order $x^{1/6}$. This is the reason for the greater intensity at the bow angle. Naturally, the field changes gradually from order $x^{1/6}$ at the bow angle to order unity somewhat away from the bow angle on the lit side. To understand this transition better, we must examine the neighborhood of the bow caustic more carefully. To do this, we expand the phase function ϕ_p in a Taylor series about the scattering angle θ_B as well as the saddle point τ_B, i.e.,

$$\phi_p = \phi_p(\theta_B,\tau_B) + \frac{\partial\phi_p}{\partial\theta}(\theta_B,\tau_B)(\theta-\theta_B) + \frac{\partial^2\phi_p}{\partial\theta\partial\tau}(\theta_B,\tau_B)(\theta-\theta_B)(\tau-\tau_B) + \ldots$$

$$+ \frac{1}{6}\frac{\partial^3\phi_p}{\partial\tau^3}(\theta_B,\tau_B)(\tau-\tau_B)^3 + \ldots, \quad (63)$$

where $\frac{\partial^3\phi_p}{\partial\tau^3}$ is given by (61) and

$$\frac{\partial^2\phi_p}{\partial\theta\partial\tau}(\theta_B,\tau_B) = \begin{cases} \pm\cos\tau_B & p = 3,5,7,\ldots \\ \mp\cos\tau_B & p = 2,4,6,\ldots, \end{cases} \quad (64)$$

with upper sign for choice A and lower sign for choice B. The first improvement

in $F_p^G(\theta)$ in the neighborhood of θ_B comes from retaining an additional term in the exponent of the integrand of the integral which led to (58). In other words we retain all the terms in the Taylor expansion of ϕ_p listed in (63) but no more. Then

$$F_p^G(\theta) \sim (-)^{p+s} x^{\frac{1}{2}} (2\pi \sin\theta)^{-\frac{1}{2}} \sin\tau_B (\cos\tau_B)^{\frac{1}{2}} R^{p-1}(\tau_B) \left[1 - R^2(\tau_B)\right]$$

$$\times \exp \left\{ ix \left[\phi_p(\theta_B, \tau_B) + \frac{\partial \phi_p}{\partial \theta}(\theta_B, \tau_B)(\theta - \theta_B) \pm \frac{\pi}{4} \right] \right\}$$

$$\times \int_{-\infty}^{\infty} \exp \left\{ ix \left[\frac{\partial^2 \phi_p}{\partial \theta \partial \tau}(\theta_B, \tau_B)(\tau - \tau_B) + \frac{\phi_p'''(\tau_B)}{6}(\tau - \tau_B)^3 \right] \right\} d\tau \ . \qquad (65)$$

The integral above is Airy's integral, equal to

$$2 \left(\frac{6}{x \phi_p'''(\tau_B)} \right)^{\frac{1}{3}} A \left[-(\theta - \theta_B) \left(\frac{6x^2}{\phi_p'''(\tau_B)} \right)^{\frac{1}{3}} \frac{\partial^2 \phi_p}{\partial \theta \partial \tau}(\theta_B, \tau_B) \right] \qquad (66)$$

where A is Airy's function, defined by

$$A(z) = \frac{1}{2} \int_{-\infty}^{\infty} \exp\left[-i(y^3 - zy)\right] dy \ . \qquad (67)$$

It derives its name from Airy's investigation of the rainbow problem over 100 years ago [19]. Substituting equations (56),(59-62), (64) and (66) into (65), we obtain

$$F_p^G(\theta) \sim (-)^{p+s} x^{\frac{1}{6}} (2\pi \sin\theta)^{-\frac{1}{2}} 2\cdot 3^{\frac{1}{3}} (N^2-1)^{\frac{1}{2}} (p^2-N^2)^{\frac{1}{12}} (p^2-1)^{-\frac{11}{12}} p^{\frac{2}{3}} \left(\frac{p-1}{p+1}\right)^{p-1} \left[1-\left(\frac{p-1}{p+1}\right)^2\right]$$

$$\times \begin{cases} \exp\left\{ix\left[2(p^2-1)^{\frac{1}{2}}(N^2-1)^{\frac{1}{2}} \pm (p^2-N^2)^{\frac{1}{2}}(p^2-1)^{-\frac{1}{2}}(\theta-\theta_B) \pm \frac{\pi}{4}\right]\right\} \\ \quad \times A\left[\mp(\theta-\theta_B)x^{\frac{2}{3}}\left\{3p^2(p^2-1)^{-2}(p^2-N^2)\right\}^{\frac{1}{3}}\right], \quad p = 3,5,7,\ldots \\ \exp\left\{ix\left[2(p^2-1)^{\frac{1}{2}}(N^2-1)^{\frac{1}{2}} \mp (p^2-N^2)^{\frac{1}{2}}(p^2-1)^{-\frac{1}{2}}(\theta-\theta_B) \pm \frac{\pi}{4}\right]\right\} \\ \quad \times A\left[\pm(\theta-\theta_B)x^{\frac{2}{3}}\left\{3p^2(p^2-1)^{-2}(p^2-N^2)\right\}^{\frac{1}{3}}\right], \quad p = 2,4,6,\ldots \end{cases} \tag{68}$$

with upper signs for choice A and lower signs for choice B. Equation (68) reduces to (58) at $\theta = \theta_B$, as it should, in view of the fact that

$$A(z) \sim \frac{1}{3}\,\frac{\sqrt{3}}{2}\,\Gamma\left(\frac{1}{3}\right) \tag{69}$$

in the neighborhood of $z = 0$. For $|z| \gg 1$, on the other hand, the Airy function decays exponentially on the shadow side, z negative, and for z positive behaves like

$$A(z) \sim \left(\frac{\pi}{3}\right)^{\frac{1}{2}} \left(\frac{3}{z}\right)^{\frac{1}{4}} \cos\left[2\left(\frac{z}{3}\right)^{\frac{3}{2}} - \frac{\pi}{4}\right] . \tag{70}$$

Therefore $F_p^G(\theta)$ loses its $x^{1/6}$ dependence and behaves like a constant on the illuminated side of the bow. This occurs when, essentially, $x^{2/3}(\theta-\theta_B)$ becomes large.

It is pertinent to inquire whether any "super-bows" exists, that is to say, whether there exists a saddle point of fourth order. This would require, in addition to the bow saddle point conditions, the vanishing of ϕ_p'''. From equation (61), it is seen that the possibility exists only for $\tau_B = \pi/2$. Therefore, from (56) and (44) or (45), we must have

$$\left.\begin{array}{l} N = p \text{ and } s = 0, \quad \theta_B = 0 \text{ , for p odd} \\ \qquad\qquad\qquad\quad \theta_B = \pi \text{ , for p even} \end{array}\right\} . \tag{71}$$

The existence of this stationary point was observed by Bucerius [18]. These directions are caustic directions, and they occur at angles for which the Laplace asymptotic forms for $Q_{\nu-1/2}(\theta)$ are not appropriate. Consequently, the super-bow field, just as the field in the neighborhood of a caustic, requires a more thorough investigation, which we shall not give here. We shall confine ourselves to the observation that a Debye saddle point evaluation of a fourth order saddle point would yield for x-dependence of the super-bow the order $x^{1/4}$. It may be conjectured that this is the correct x-dependence because experience with a similar (inappropriate) evaluation of the forward scattering from impenetrable spheres indicates that the x-dependence is correctly given, although the coefficient is not.

VII. The Diffracted Field

We will now evaluate the contribution of those integrals which we have labeled the diffracted field. From the standpoint of a stationary phase evaluation, these integrals have no stationary points. Therefore, it should not be surprising that it turns out that these terms are exponentially small compared to the geometric optics field.

It is to be kept in mind that the diffracted field as given by $F_p^D(\theta)$ is applicable only to those regions where the corresponding field $F_p^G(\theta)$ is valid, that is to say, the illuminated regions. Although the evaluation of the diffracted field for the shadow regions would be essentially unchanged, additional contributions to the diffracted field would arise from those integrals labeled geometric optics field.

The method of evaluation of a typical integral occurring in the diffracted field is to deform the path of integration along the entire ν-axis into an infinite semi-circle in the upper half of the complex ν-plane. The factors of $e^{i\pi\nu}$ in the integrand are sufficient to insure the vanishing of the integral along the semi-circle. Therefore, the integral is equal to $2\pi i$ times the sum of the residues of the integrand lying in the upper halp of the ν-plane. These residues have been investigated by Beckmann and Franz [20] in their work on the dielectric cylinder. The residues are due only to the zeros of the function L defined in equation (11). These zeros, lying in the upper half of the ν-plane, are given by the expression

$$\nu_j = x + q_j\left(\frac{x}{6}\right)^{\frac{1}{3}} e^{i\frac{\pi}{3}} + \cdots \quad , \tag{72}$$

where q_j are numbers which are determined, in the first approximation, by the equation

$$e^{-i\frac{\pi}{3}}\left(\frac{6}{x}\right)^{\frac{1}{3}}\frac{A'(q_j)}{A(q_j)} = i\,N\sin\tau_1 \ . \tag{73}$$

A more complete discussion may be found in the paper of Beckmann and Franz (l.c.).

Let us consider first $F_p^D(\theta)$ as given by (31A,B) or (33A,B). Due to the amplitude factor $T_{12}R_{22}^{p-1}T_{21}$ which appears in $f_{p,m}$, there is a (p+1)-fold root of the denominator at $\nu = \nu_j$. Consequently, the residue sum for $F_p^D(\theta)$ is

$$F_p^D(\theta) = \frac{32}{\pi x^3}(-)^{\frac{p+1}{2}+s}\sum_{j=1}^{\infty}\left\{\sum_{l=0}^{s-1}\left(\frac{\partial^p}{\partial\nu^p}\,\frac{\nu V^{p-1}}{H_\nu^{(1)}(x)H_\nu^{(2)}(x)H_\nu^{(1)}(Nx)H_\nu^{(2)}(Nx)}\right.\right.$$

$$\left.\times\frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)}\left[\frac{H_\nu^{(1)}(Nx)}{H_\nu^{(2)}(Nx)}\right]^p e^{i(p-2s)\pi\nu}P_{\nu-1/2}(-\cos\theta)e^{il2\pi\nu}\right)\left(\frac{\partial L}{\partial\nu}\right)^{-p-1}$$

$$+\sum_{m=0}^{\infty}\left(\frac{\partial^p}{\partial\nu^p}\,\frac{\nu V^{p-1}}{H_\nu^{(1)}(x)H_\nu^{(2)}(x)H_\nu^{(1)}(Nx)H_\nu^{(2)}(Nx)}\,\frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)}\left[\frac{H_\nu^{(1)}(Nx)}{H_\nu^{(2)}(Nx)}\right]^p\right.$$

$$\left.\left.\times\ e^{i(p-2s)\pi\nu}\,e^{\mp i\pi\nu}\,P_{\nu-1/2}(\cos\theta)\,e^{im2\pi\nu}\right)\left(\frac{\partial L}{\partial\nu}\right)^{-p-1}\right\}_{\nu=\nu_j},$$

$$p = 1,3,5,\ldots\ , \tag{74}$$

$$F_p^D(\theta) = \frac{32}{\pi x^3}(-)^{\frac{p}{2}+s+1} \sum_{j=1}^{\infty} \left\{ \sum_{\ell=0}^{s-1} \left(\frac{\partial^p}{\partial \nu^p} \frac{\nu V^{p-1}}{H_\nu^{(1)}(x)H_\nu^{(2)}(x)H_\nu^{(1)}(Nx)H_\nu^{(2)}(Nx)} \right.\right.$$

$$\times \frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)} \left[\frac{H_\nu^{(1)}(Nx)}{H_\nu^{(2)}(Nx)}\right]^p e^{i(p-2s)\pi\nu} P_{\nu-1/2}(\cos\theta) e^{i\ell 2\pi\nu} \left.\right) \left(\frac{\partial L}{\partial \nu}\right)^{-p-1}$$

$$\pm \sum_{m=0}^{\infty} \left(\frac{\partial^p}{\partial \nu^p} \frac{\nu V^{p-1}}{H_\nu^{(1)}(x)H_\nu^{(2)}(x)H_\nu^{(1)}(Nx)H_\nu^{(2)}(Nx)} \frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)} \left[\frac{H_\nu^{(1)}(Nx)}{H_\nu^{(2)}(Nx)}\right]^p \right.$$

$$\left.\left. \times e^{i(p-2s)\pi\nu} e^{\pm i\pi\nu} P_{\nu-1/2}(-\cos\theta) e^{im2\pi\nu} \right) \left(\frac{\partial L}{\partial \nu}\right)^{-p-1} \right\}_{\nu=\nu_j} ,$$

$$p = 2,4,6,\ldots , \qquad (75)$$

with upper signs for choice A, and lower signs for choice B. The quantities V and L are defined in equations (11). Of course, the value of s in these equations is already prescribed by the associated geometric optics field $F_p^G(\theta)$.

These terms have been labeled "creeping waves" [7] or "diffracted rays" [8] because they are just what would be contributed by rays which encircle the sphere, decay exponentially, and shed rays tangentially as they move along. This can be seen by examining the phase factors appearing in (74) and (75). With the appropriate asymptotic formulas inserted, the exponential factors in the summation over m are

$$\exp\left\{ix(2pN\sin\tau_1 - 2s\sin\tau) + i\nu\left[(p-2s)\pi \mp \pi + 2\tau - 2p\tau_1 \pm \theta + 2\pi m\right]\right\}_{\nu=\nu_j}, \quad p = 1,3,5,\ldots, \tag{76}$$

$$\exp\left\{ix(2pN\sin\tau_1 - 2s\sin\tau) + i\nu\left[(p-2s)\pi \mp \pi + 2\tau - 2p\tau_1 \pm (\pi-\theta) + 2\pi m\right]\right\}_{\nu=\nu_j}, \quad p = 2,4,6,\ldots, \tag{77}$$

where the choices in sign are independent. The $\mp\pi$ arises from the A or B selection. The $\pm\theta$ and $\pm(\pi-\theta)$ are due to the two possible exponentials in the asymptotic forms of $P_{\nu-1/2}(\cos\theta)$ and $P_{\nu-1/2}(-\cos\theta)$, respectively. Similar expressions are contained in the summation over ℓ. From equation (72), it can be seen that at $\nu = \nu_j$, $\tau \sim 0$ and $\tau_1 \sim \cos^{-1}\frac{1}{N}$. Consequently, the diffracted field contribution arises from the grazing incident ray as was first pointed out by Keller [10] in the application of geometric diffraction theory to infinitely conducting obstacles. The first term in the brackets describes the phase factor caused by the p transmissions through the interior of the sphere. From the second term, we see that these terms are exponentially small, because ν_j has a real imaginary part. The exponential decay factors are

$$\exp\left\{-q_j\left(\frac{x}{6}\right)^{\frac{1}{3}}\cos\frac{\pi}{6}\left[2p\sin^{-1}\frac{1}{N} \mp \pi - 2s\pi \pm \theta + 2\pi m\right]\right\}, \quad p = 1,3,5,\ldots, \tag{78}$$

$$\exp\left\{-q_j\left(\frac{x}{6}\right)^{\frac{1}{3}}\cos\frac{\pi}{6}\left[2p\sin^{-1}\frac{1}{N} \mp \pi - 2s\pi \pm (\pi-\theta) + 2\pi m\right]\right\}, \quad p = 2,4,6,\ldots. \tag{79}$$

The quantity in the square brackets is precisely that angle of encirclement about the sphere necessary to permit a tangential ray to be shed in the θ-direction,

where θ is an angle in the lit region. It is to be kept in mind that the parameter s was chosen so as to insure this. In contract, the quantity $2\pi m$ describes the additional decay due to m-fold encirclement of the sphere. The $\pm$ signs in front of the quantities θ and $(\pi-\theta)$ that occur in (78) and (79), which have their origin in the two exponentials appearing in the Laplace asymptotic forms of $P_{\nu-1/2}(\pm\cos\theta)$, describe the two possible ways for diffracted rays to be shed in the θ direction. This is illustrated in figures 5 and 6 for $m = 0$, $p = 1,2$, and $s = 0$.

The diffracted field $F_o^D(\theta)$ which is associated with the reflected ray contains a sum which is independent of R_{11}, according to (27). This sum may be shown to vanish. This follows because, on the one hand, the imaginary part of the integral in the summand is an even function of ν while the real part is odd. Therefore, the integral over the entire ν axis is pure imaginary and equal to twice the value of the integral from zero to infinity. On the other hand, the latter integral has a null imaginary part. This may be seen by deforming the path of integration to the positive imaginary ν-axis. In view of the fact that $P_{i\nu-1/2}(-\cos\theta)$ is a real function, it follows that the integral is a real function.

Therefore, the diffracted field $F_o^D(\theta)$ consists only of the second sum in (27). This sum may be treated in the same manner as $F_p^D(\theta)$. We find for it that

$$F_o^D(\theta) = \frac{2\pi i}{x} \sum_{m=o}^{\infty} (-)^m \sum_{j=1}^{\infty} \left\{ \frac{\nu U}{\left(\frac{\partial L}{\partial \nu}\right)} \frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)} e^{i\pi\nu} P_{\nu-1/2}(-\cos\theta) e^{i2\pi m\nu} \right\}_{\nu=\nu_j} . \tag{80}$$

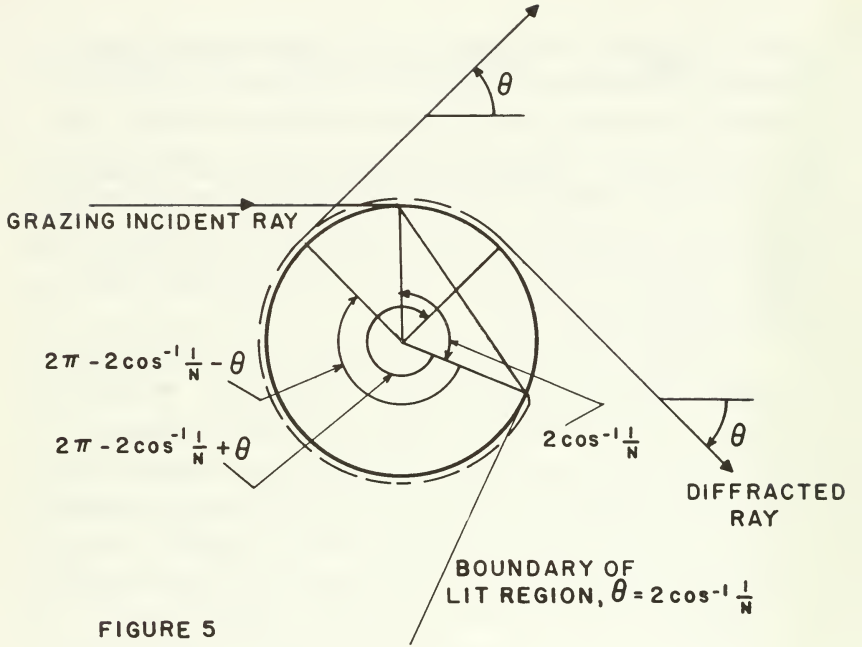

FIGURE 5

The diffracted rays which are shed in the direction θ of the lit region for p = 1 and m = 0 are given by $F_1^D(\theta)$, equation (74) with the selection (0,A). There are two different possible angles of encirclement of the sphere which yield a diffracted ray in the direction θ. They correspond to the two possible exponentials appearing in the asymptotic form of $P_{\nu-1/2}(\cos\theta)$. The angles of encirclement are indicated by equation (78) with choice A. Only the upper grazing-incident ray is shown.

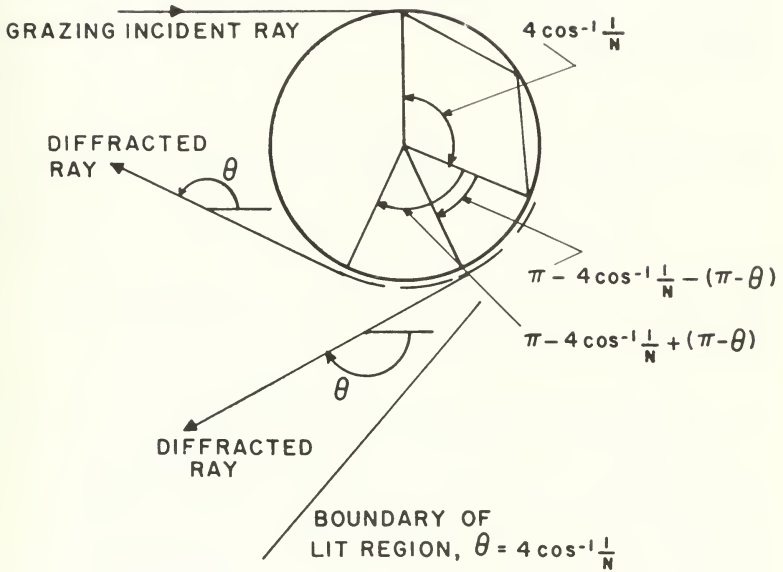

FIGURE 6

The diffracted rays shed in the direction θ of the lit region associated with $p = 2$ and $m = 0$ are expressed by $F_2^D(\theta)$, equation (75). The figure illustrates the selection (0,A). If $4\cos^{-1}\frac{1}{N}$ is greater than π, the selection (0,B) would have to be made. The decay factors corresponding to the figure are given by equation (79).

This expression applies to the lit region for the reflected field which, as we have seen, extends everywhere except the two caustic directions $\theta = 0$ and $\theta = \pi$. It contains a decay factor given by equation (79) with $p = s = 0$ and selection A. If we pass to the limit of the infinitely conducting sphere, equation (80) reduces to

$$F_o^D(\theta) = \frac{2\pi i}{x} \sum_{m=o}^{\infty} (-)^m \sum_{j=1}^{\infty} \left\{ \nu \frac{H_\nu^{(2)}(x)}{\partial H_\nu^{(1)}(x)/\partial \nu} e^{i\pi\nu} P_{\nu-1/2}(-\cos\theta) e^{i2\pi m\nu} \right\}_{\nu=\nu_j} . \tag{81}$$

The zeros ν_j are the zeros of $H_\nu^{(1)}(x)$ in this case. This agrees with the result of Franz [6] and Levy and Keller [9]

VIII. Axial Caustics and Boundaries of Illuminated Regions.

The results we have obtained are not valid in the neighborhood of the axial caustics, that is to say, the forward and backward scattering directions, and in the neighborhood of the boundaries of the illuminated regions for each of the partial amplitudes $F_p(\theta)$. In the latter case, this is because the boundary is determined by the saddle point value $\tau_o = 0$, and the Debye asymptotic formulas are not valid at that point. This state of affairs can in fact be easily corrected with the use of the uniform asymptotic formulas for Hankel functions in terms of Airy functions. While the breakdown of the Debye formulas may also contribute to the lack of validity of our results in the neighborhood of the axial directions, the more serious cause is the infinite behavior of the generalized Legendre functions $Q_\nu^{(1),(2)}(\theta)$ in the forward and backward directions. In this case, it appears necessary to

rewrite the expressions for $F_p(\theta)$ so that they are valid in the neighborhood of these directions. This will be investigated in a subsequent paper.

Acknowledgement

The author is happy to take this opportunity to express his indebtedness to Joseph B. Keller for a number of stimulating and illuminating conversations about this work.

Appendix

Useful Formulas

In connection with Legendre functions, it is convenient to define $Q_\nu^{(1)}(\theta)$ and $Q_\nu^{(2)}(\theta)$ in the following way so that these functions have purely exponential asymptotic forms

$$Q_\nu^{(1)}(\theta) = \frac{1}{2}\left[P_\nu(\cos\theta) + i\,\frac{2}{\pi}\,Q_\nu(\cos\theta)\right], \tag{82}$$

$$Q_\nu^{(2)}(\theta) = \frac{1}{2}\left[P_\nu(\cos\theta) - i\,\frac{2}{\pi}\,Q_\nu(\cos\theta)\right], \tag{83}$$

Here, $Q_\nu(\cos\theta)$ is the generalized Legendre function of the second kind. Then the relation between Legendre functions may be written as

$$P_\nu(-\cos\theta) = e^{-i\pi\nu}\,P_\nu(\cos\theta) + 2i\,\sin\pi\nu\,Q_\nu^{(1)}(\theta), \tag{84}$$

$$P_\nu(-\cos\theta) = e^{i\pi\nu}\,P_\nu(\cos\theta) - 2i\,\sin\pi\nu\,Q_\nu^{(2)}(\theta). \tag{85}$$

The Laplace asymptotic forms for Legendre functions [21] are

$$P_\nu(\cos\theta) \sim \left(\frac{1}{2}\pi\nu\,\sin\theta\right)^{-\frac{1}{2}}\cos\left[(\nu+\tfrac{1}{2})\theta - \tfrac{\pi}{4}\right],\quad 0<\theta<\pi, \tag{86}$$

$$\frac{2}{\pi}\,Q_\nu(\cos\theta) \sim -\left(\frac{1}{2}\pi\nu\,\sin\theta\right)^{-\frac{1}{2}}\sin\left[(\nu+\tfrac{1}{2})\theta - \tfrac{\pi}{4}\right],\quad 0<\theta<\pi. \tag{87}$$

With the use of these expressions, the asymptotic forms of $Q_\nu^{(1),(2)}(\theta)$ become

$$Q_\nu^{(1)}(\theta) \sim (2\pi\nu \sin\theta)^{-\frac{1}{2}} \exp\left[-i(\nu + \tfrac{1}{2})\theta + i\tfrac{\pi}{4}\right], \quad 0 < \theta < \pi \ , \tag{88}$$

$$Q_\nu^{(2)}(\theta) \sim (2\pi\nu \sin\theta)^{-\frac{1}{2}} \exp\left[+i(\nu + \tfrac{1}{2})\theta - i\tfrac{\pi}{4}\right], \quad 0 < \theta < \pi \ . \tag{89}$$

The Debye asymptotic forms for the Hankel functions [22] and their derivatives, for large argument x and index $\nu \ll x$, are

$$H_\nu^{(1)}(x) \sim (\tfrac{1}{2}\pi x \sin\tau)^{-\frac{1}{2}} \exp\left[ix(\sin\tau - \tau\cos\tau) - i\tfrac{\pi}{4}\right], \tag{90}$$

$$H_\nu^{(2)}(x) \sim (\tfrac{1}{2}\pi x \sin\tau)^{-\frac{1}{2}} \exp\left[-ix(\sin\tau - \tau\cos\tau) + i\tfrac{\pi}{4}\right], \tag{91}$$

$$\frac{dH_\nu^{(1)}(x)}{dx} \sim i\sin\tau(\tfrac{1}{2}\pi x \sin\tau)^{-\frac{1}{2}} \exp\left[ix(\sin\tau - \tau\cos\tau) - i\tfrac{\pi}{4}\right], \tag{92}$$

$$\frac{dH_\nu^{(2)}(x)}{dx} \sim -i\sin\tau(\tfrac{1}{2}\pi x \sin\tau)^{-\frac{1}{2}} \exp\left[-ix(\sin\tau - \tau\cos\tau) + i\tfrac{\pi}{4}\right], \tag{93}$$

where $\tau = \cos^{-1}\frac{\nu}{x}$. (94)

According to the method of steepest descents [23], if the integral $\mathcal{Q}$ to be evaluated asymptotically for $\lambda \gg 1$ is of the form

$$\mathcal{Q} = \int A(\xi) e^{i\lambda\phi(\xi)} d\xi \tag{95}$$

and contains an interior point along its path of integration such that

$$\frac{d\phi}{d\xi} = 0, \quad \frac{d^2\phi}{d\xi^2} = 0, \ldots, \quad \phi^{(m)} = \frac{d^m\phi}{d\xi^m} \neq 0 \qquad \text{at } \xi = \xi_o , \tag{96}$$

then it is said to have a saddle point of order m at $\xi = \xi_o$. If ϕ and A are expanded in Taylor series about the point $\xi = \xi_o$, e.g.,

$$\phi = \phi(\xi_o) + \frac{1}{m!}\phi^{(m)}(\xi_o)(\xi-\xi_o)^m + \ldots , \tag{97}$$

and A is of the form

$$A = A_o + A_1(\xi - \xi_o) + \ldots , \tag{98}$$

then the dominant term (qua function of λ) in the asymptotic expansion of $\mathcal{Q}$ is given by

$$\mathcal{Q} \sim A_o\, e^{i\lambda\phi(\xi_o)} \left[\frac{2\pi}{\lambda\phi''(\xi_o)}\right]^{\frac{1}{2}} e^{i\frac{\pi}{4}} \qquad \text{if } m = 2 , \tag{99}$$

and

$$\mathcal{Q} \sim 3^{-\frac{1}{2}} A_o\, e^{i\lambda\phi(\xi_o)} \left[\frac{6}{\lambda\phi'''(\xi_o)}\right]^{\frac{1}{3}} \Gamma(\tfrac{1}{3}) \qquad \text{if } m = 3 . \tag{100}$$

References

[1] H.C. van de Hulst, Light Scattering by Small Particles, John Wiley and Sons, New York, 1957.

[2] S.I. Rubinow, On the Forward Scattering from a Dielectric Sphere at High Frequencies, Scientific Report No. 7, Cruft Laboratory, Harvard University (1956).

[3] S.I. Rubinow and T.T. Wu, J. Appl. Phys. 27, 1032 (1956); Scientific Report No. 3, Cruft Laboratory, Harvard University (1955).

[4] T.T. Wu, Phys. Rev. 104, 1201 (1956); Technical Report No. 232, Cruft Laboratory, Harvard University (1956).

[5] P. Debye, Phys. Zeit. 9, 775 (1908).

[6] W. Franz, Zeit. f. Naturf. 9a, 705 (1954).

[7] W. Franz and K. Deppermann, Ann. Physik 10, 361 (1952); Ann. Physik 14, 253 (1954).

[8] J.B. Keller, Symposium on Microwave Optics, Eaton Electronics Research Laboratory, McGill University, Montreal, Canada (1953); Proceedings of Symposia in Applied Mathematics, Vol. 8, 27, McGraw-Hill, New York, 1958.

[9] B.R. Levy and J.B. Keller, Comm. Pure and Appl. Math. 12, 159 (1959).

[10] J.B. Keller, URSI Michigan Symposium on Electromagnetic Wave Propagation, Ann Arbor (1955); IRE - Transactions on Antennas and Propagation AP-4, No. 3, 312 (1956).

[11] L.I. Schiff, Quantum Mechanics, McGraw-Hill, New York, 1949, Chap. V.

[12] G.N. Watson, Proc. Roy. Soc. A, 95, 83 (1919).

[13] P.M. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw-Hill, New York, 1953, Vol. I, p. 466.

[14] B. van der Pol and H. Bremmer, Phil. Mag. 24, 141, 825 (1937).

[15] H. Bremmer, Terrestrial Radio Waves, Elsevier Publishing Co., New York, 1949.

[16] J.B. Keller, R.M. Lewis, and B.D. Seckler, Comm. Pure. Appl. Math. 9, 207 (1956).

[17] H. Goldstein, Classical Mechanics, Addison-Wesley, Cambridge, 1950, p. 92.

[18] H. Bucerius, Optik 1, 188 (1946).

[19] G.B. Airy, Proc. Cambridge Phil. Soc. 6, 379 (1839).

[20] P. Beckmann and W. Franz, Zeit. f. Naturf. 12a, 257 (1957).

[21] A. Erdélyi, Higher Transcendental Functions, Vol. I, McGraw-Hill, New York, 1953, p. 162.

[22] A. Sommerfeld, Partial Differential Equations in Physics, Academic Press, New York, 1949, p. 119.

[23] L. Brillouin, Ann. de L'École Normales Supérieure, 3.33, 27 (1916).

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